(NASA-CR-93619) ANALYTICAL SOLUTION TO AN OPTIMUM TWO BURN DEBOOST INTO PARKING ORBIT (Bellcomm, Inc.) 27 p

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SUBJECT: Analytical Solution to an Optimum Two Burn Deboost Into Parking Orbit - Case 310

DATE: February 14, 1968

FROM: S. F. Caldwell

MEMORANDUM FOR FILE

[redacted]

INTRODUCTION

Under the existing Apollo flight plan a single burn deboost is used to place the spacecraft into a lunar parking orbit which passes over the landing site. Prior memorandums (Reference 1 and 2) have proposed a two burn deboost, having one burn to perform a partial plane change and establish a circular parking orbit and a second burn to complete the plane change. A two burn deboost can be used to either maximize lunar accessible area for a fixed ΔV or to minimize the total ΔV required to reach a given lunar parking orbit. In the referenced studies a limited number of computer runs were made from which the empirically optimum solution was chosen. This memorandum advances methods for solving both cases analytically.

ORBITAL MECHANICS

Figure 1 shows a schematic sketch of a two burn deboost. From a hyperbola in the initial plane, the α-plane,* the CSM deboosts into a circular parking orbit in the β_1-plane performing a plane change of ρ_1 with a velocity change of ΔV_1. After a flight of ϕ degrees in the β_1-plane a second burn is made to place the spacecraft in a circular parking orbit in the β_2-plane, after a plane change of ρ_2 with a velocity change of ΔV_2. The inclination of the β_2-plane above the α-plane is denoted as i_2. λ is the arc distance in the α-plane between perilune on the hyperbola and the node of the β_2-plane and α-plane. The true anomaly of the hyperbolic deboost point is denoted as f. Figures 2 and 3 define ΔV_1 and ΔV_2 in terms of circular orbital velocity (V_C), hyperbolic velocity at deboost (V_H) and the angles ρ_1, ρ_2, and γ, the flight path angle at the first burn.

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*Table I contains a glossary of symbols and definitions.

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ABSTRACT

Under the existing Apollo flight plan a single burn deboost, providing a combined velocity and plane change, is used to place the spacecraft into a lunar parking orbit which passes over the landing site. In the "two burn deboost", an additional burn is performed in lunar parking orbit before LM descent for the purpose of adjusting the orbit plane. Proper combinations of plane changes on the first and second burns will, in general, result in lower SM propellant costs for the two burn deboost. While utilizing multiple lunar orbit insertion burns, the technique gives these propellant savings without sacrificing any of the advantages of the free return flight plan.

This memorandum describes an efficient method for analytically calculating the optimum combination of the plane changes for the two burn case by an iterative technique. Methods are presented for solving the two basic mission analysis problems: (1) the calculation of the maneuvers required to reach a specified lunar parking orbit (passing over the lunar landing site) with minimum propellant costs and (2) the calculation of the maximum lunar area accessible for a fixed propellant budget. Quantitative results are presented which show that the greatest savings are realized for cases wherein the parking orbit/approach hyperbola node is more than 10° to 15° from hyperbolic perilune; such geometry arises most frequently for lunar landing sites in the central region.

Modifications to existing targeting programs in order to accommodate this technique are believed to be small. No convergence difficulties have been identified with the iterative technique that has been employed.

Figures 1, 2, and 3 assume an off-perilune deboost into parking orbit. Off-perilune deboosts are used in the case of a single burn to move the line of nodes between the hyperbolic plane and the final parking orbit plane. Such movement of the node results in a geometric efficiency which can be used to minimize propellant requirements even though the flight path angle at deboost is non-optimum. For this two burn deboost, an off-perilune maneuver is shown to also result in reduced propellant requirements.

THE MINIMUM ΔV_T PROBLEM

When minimizing the total ΔV required to reach a specified lunar orbit plane from the specified hyperbola, the β_2-plane is fixed and hence λ and i_2 are known. The β_1-plane which results in minimum total ΔV must be determined. The β_1-plane can be completely specified by two independent parameters, the node f and the inclination ρ_1 relative to the hyperbola plane (α-plane). Hence the geometry of the solution is completely specified by the two parameters f and ρ_1 when λ and i_2 are known. One may express this relationship functionally as

$$\Delta V_T = F_1 (f, \rho_1). \tag{1}$$

Since ρ_1 and f are independent the necessary conditions that ΔV_T possess a minimum or a maximum are

$$\frac{\partial F_1}{\partial \rho_1} = 0 \text{ and } \frac{\partial F_1}{\partial f} = 0. \qquad (2,\ 3)$$

For algebraic convenience the problem has been formulated in terms of seven variables ΔV_T, ΔV_1, ΔV_2, ρ_1, ρ_2, f and ϕ; λ and i_2 are specified. The plan of attack will be to obtain relationships among these variables which allow the analytic determination of $\partial F_1/\partial \rho_1$ and $\partial F_1/\partial f$; an additional expression for ΔV_T in terms of ρ_1, ρ_2, f and ϕ is required.

Starting with an equation for total ΔV,

$$\Delta V_T = \Delta V_1 + \Delta V_2, \tag{4}$$

then from Figure 2 and the law of cosines

$$\Delta V_1^2 = V_C^2 + V_H^2 - 2V_C V_H \cos \rho_1 \cos \gamma, \tag{5}$$

where

$$\tan \gamma = \frac{e \sin f}{1 + e \cos f}, \tag{5a}$$

$$e = \frac{-r \cos f - \sqrt{(r \cos f)^2 - 4a(r-a)}}{2a} \qquad (5b)^*$$

and

$$a = \frac{-\mu}{V_H^2 - 2\frac{\mu}{r}} \tag{5c}$$

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*Equation (5b) results from the solution of the conic equation, $r = \dfrac{a(1 - e^2)}{1 + e \cos f}$. The choice of the negative root in the quadratic solution is unambiguous since the positive root would result in negative eccentricities.

In the above equations μ is the gravitational constant of the moon and r is the radius of the lunar parking orbit. From Figure 3,

$$\Delta V_2 = 2V_c \sin(\rho_2/2). \tag{6}$$

From the law of sines

$$\frac{\sin \rho_2}{\sin(\lambda - f)} = \frac{\sin i_2}{\sin \phi}, \text{ or}$$

$$\sin \rho_2 \sin \phi = \sin i_2 \sin(\lambda - f). \tag{7}$$

A fifth equation can be obtained from Figure 1 and a law of spherical trigonometry (Reference 3, p. 12).

$$\cos(\lambda - f) \cos \rho_1 = \sin(\lambda - f) \cot \phi + \sin \rho_1 \cot i_2 \tag{8}$$

The required sixth and seventh equations are provided by the conditions imposed by equations 2 and 3 which can now be expressed in terms of the variables ΔV_T, ΔV_1, ΔV_2, ρ_1, ρ_2, f, ϕ, i_2 and λ. Equations (9) through (21) develop explicit partial derivatives for $\partial \Delta V_T/\partial \rho_1$ (Equation 13) and $\partial \Delta V_T/\partial f$ (Equation 21) by successive differentiation of Equations (4) through (8) and appropriate algebraic substitution. An expression for the partial derivative of ΔV_T with respect to ρ_1 may be formed by taking the partial of both sides of Equation (4).

$$\frac{\partial \Delta V_T}{\partial \rho_1} = \frac{\partial \Delta V_1}{\partial \rho_1} + \frac{\partial \Delta V_2}{\partial \rho_1} \tag{9}$$

The partial of Equation (5) divided by $2\Delta V_1$ and the partial of (6) are

$$\frac{\partial \Delta V_1}{\partial \rho_1} = \frac{V_C V_H}{\Delta V_1} \sin \rho_1 \cos \gamma$$

$$\frac{\partial \Delta V_2}{\partial \rho_1} = V_C \cos (\rho_2/2) \frac{\partial \rho_2}{\partial \rho_1} \quad .$$

Substituting these equations in (9) and dividing through by V_C gives

$$\frac{1}{V_C} \frac{\partial \Delta V_T}{\partial \rho_1} = \frac{V_H \sin \rho_1 \cos \gamma}{\Delta V_1} + \cos (\rho_2/2) \frac{\partial \rho_2}{\partial \rho_1} \quad . \tag{10}$$

Taking the partial derivative of Equation (7) with respect to ρ_1,

$$\sin \phi \cos \rho_2 \frac{\partial \rho_2}{\partial \rho_1} + \sin \rho_2 \cos \phi \frac{\partial \phi}{\partial \rho_1} = 0,$$

and solving for $\partial\rho_2/\partial\rho_1$,

$$\frac{\partial \rho_2}{\partial \rho_1} = - \tan \rho_2 \cot \phi \frac{\partial \phi}{\partial \rho_1} \ , \tag{11}$$

gives an expression for the partial of ρ_2 with respect to ρ_1. In order to calculate $\partial\phi/\partial\rho_1$, it is necessary to take the partial derivative of Equation (8) with respect to ρ_1,

$$- \cos(\lambda-f) \sin\rho_1 = - \sin(\lambda-f) \csc^2\phi \frac{\partial\phi}{\partial\rho_1} + \cot i_2 \cos\rho_1{}^{*}.$$

Solving for $\partial\phi/\partial\rho_1$,

$$\frac{\partial\phi}{\partial\rho_1} = \frac{\cos\rho_1 \cot i_2 + \cos(\lambda-f)\sin\rho_1}{\sin(\lambda-f)\csc^2\phi} , \qquad (12)$$

gives an expression for the partial of ϕ with respect to ρ_1. Substituting (12) into (11) and then substituting (11) back into (10) gives an explicit expression for $\partial\Delta V_T/\partial\rho_1$, the sixth required equation:

$$\frac{1}{V_C} \frac{\partial\Delta V_T}{\partial\rho_1} = 0 = \frac{V_H \sin\rho_1 \cos\gamma}{\Delta V_1} - \qquad (13)$$

$$\frac{\cos(\rho_2/2)\cos\phi\tan\rho_2 (\cos\rho_1 \cot i_2 + \cos(\lambda-f)\sin\rho_1)}{\csc\phi \sin(\lambda-f)} .$$

The next step is to develop the seventh equation, $\partial\Delta V_T/\partial f$, by taking the partial of Equation (4) with respect to f,

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*Note that this partial derivative is obtained by observing that ρ_1 and f are independent; hence $\partial f/\partial\rho_1 = 0$.

$$\frac{\partial \Delta V_T}{\partial f} = \frac{\partial \Delta V_1}{\partial f} + \frac{\partial \Delta V_2}{\partial f} \ . \tag{14}$$

The first term in Equation (14) is obtained by taking the partial of Equation (5) divided by $2\Delta V_1$,

$$\frac{\partial \Delta V_1}{\partial f} = \frac{V_C V_H}{\Delta V_1} \cos \rho_1 \sin \gamma \frac{\partial \gamma}{\partial f} \ . \tag{15}$$

The partial of Equation (5a) with respect to f yields

$$\frac{1}{\cos^2 \gamma} \frac{\partial \gamma}{\partial f} = \frac{e^2 + e \cos f + \sin f \frac{\partial e}{\partial f}}{(1 + e \cos f)^2} \ .$$

Multiplying by $\cos^2 \gamma$ and using Equation (5a)

$$\frac{\partial \gamma}{\partial f} = \frac{\sin^2 \gamma}{\sin^2 f} \left[1 + \frac{\cos f}{e} + \frac{\sin f}{e^2} \frac{\partial e}{\partial f} \right] . \tag{16}$$

The remaining unknown, $\partial e/\partial f$, is determined by taking the partials of Equation (5b),

$$\frac{\partial e}{\partial f} = \frac{r \sin f}{2a} + \frac{r^2 \cos f \sin f}{2a \sqrt{r^2 \cos^2 f - 4a\,(r-a)}} \ . \tag{17}$$

Substituting Equations (17) and (16) into (15) then gives an explicit equation for $\partial \Delta V_1/\partial f$. One can now proceed to a determination of the second term in Equation (14) by taking the partial derivative of Equation (6) with respect to f,

$$\frac{\partial \Delta V_2}{\partial f} = V_C \cos (\rho_2/2) \frac{\partial \rho_2}{\partial f} . \tag{18}$$

Taking the partial of Equation (7) with respect to f,

$$\cos \rho_2 \sin \phi \frac{\partial \rho_2}{\partial f} + \sin \rho_2 \cos \phi \frac{\partial \phi}{\partial f} = - \sin i_2 \cos (\lambda - f) ,$$

and solving for $\partial \rho_2/\partial f$,

$$\frac{\partial \rho_2}{\partial f} = - \frac{\sin i_2 \cos (\lambda - f) + \sin \rho_2 \cos \phi \frac{\partial \phi}{\partial f}}{\cos \rho_2 \sin \phi} , \tag{19}$$

gives an expression for the partial of ρ_2 with respect to f. The partial of Equation (8) with respect to f,

$$\sin (\lambda - f) \cos \rho_1 = - \csc^2 \phi \sin (\lambda - f) \frac{\partial \phi}{\partial f} - \cot \phi \cos (\lambda - f) ,$$

can be reformulated to give an expression for $\partial \phi/\partial f$,

$$\frac{\partial \phi}{\partial f} = - \sin^2 \phi \cos \rho_1 - \cos \phi \sin \phi \cot (\lambda - f) . \tag{20}$$

Substituting (20) and (19) into (18) gives $\partial \Delta V_2/\partial f$; then substituting (18) and (15) into (14) gives an expression for the partial of ΔV_T with respect to f:

$$\frac{1}{V_C}\frac{\partial \Delta V_T}{\partial f} = \frac{V_H}{\Delta V_1} \cos \rho_1 \frac{\sin^3 \gamma}{\sin^2 f} \left[1 + \frac{\cos f}{e} + \frac{\sin f}{e^2} \left\{ \frac{r \sin f}{2a} + \frac{r^2 \cos f \sin f}{2a \sqrt{r^2 \cos^2 f - 4a (r-a)}} \right\} \right] + \cos (\rho_2/2) \left[- \frac{\sin i_2 \cos (\lambda - f)}{\cos \rho_2 \sin \phi} + \tan \rho_2 \cos \phi (\sin \phi \cos \rho_1 + \cos \phi \cot (\lambda - f)) \right] = 0. \qquad (21)$$

Equations (13) and (21) then along with Equations (4) through (8) are sufficient to uniquely determine the seven unknowns: ΔV_T, ΔV_1, ΔV_2, ρ_1, ρ_2, f, and ϕ. Knowing λ and i_2 and making initial assumptions for ρ_1 and f, ϕ can be found from Equation (8)

$$\cot \phi = \frac{\cos (\lambda - f) \cos \rho_1 - \sin \rho_1 \cot i_2}{\sin (\lambda - f)}$$

or inverting

$$\phi = \arctan \left[\frac{\sin (\lambda - f)}{\cos (\lambda - f) \cos \rho_1 - \sin \rho_1 \cot i_2} \right], \quad 0° \leq \phi \leq 180° \qquad (22)$$

and ρ_2 can be calculated from Equation (7)

$$\rho_2 = \arcsin \left(\frac{\sin i_2 \sin (\lambda - f)}{\sin \phi} \right) \qquad (23)$$

and the two burn maneuver required to proceed from the α-plane to the β_2-plane is determined.

Having these values, ΔV_1, ΔV_2, and ΔV_T can be calculated from Equations (4) through (6) and the examination for optimality can be made.

Substituting values for ΔV_1, ρ_1, ρ_2, f, and ϕ back in Equation (13) and Equation (21), values for $\partial \Delta V_T/\partial \rho_1$ and $\partial \Delta V_T/\partial f$ can be obtained. If the partial of ΔV_T with respect to ρ_1 (or f) turns out to be positive, ρ_1 (or f) is too large; if it is negative, ρ_1 (or f) is too small. Using this information and successive calculations, ρ_1 and f can be found such that the partials of ΔV_T with respect to ρ_1 and f are sufficiently close to zero. It should be noted that this process inherently approaches the minimum of ΔV_T rather than the maximum.

One recursive technique, that has proven to converge rapidly, simply assumes ΔV_T is a second order function of both ρ_1 and f. Starting with a value of ρ_1 and two initial guesses for f (f_1 and f_2) and calculating the partials of ΔV_T with respect to f at both f_1 and f_2, the slope of the line passing through these two points can be calculated*,

$$\frac{\dfrac{\partial \Delta V_T}{\partial f_1} - \dfrac{\partial \Delta V_T}{\partial f_2}}{f_1 - f_2} .$$

This slope can be used to calculate a value of f for which the value of $\partial \Delta V_T/\partial f$ would be zero if $\partial \Delta V_T/\partial f$ were truly linear in f:

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*For notational simplicity, the partial derivative of ΔV_T with respect to f at $f = f_i$ will be expressed as $\partial \Delta V_T/\partial f_i$.

$$f_3 = f_2 - \frac{\partial \Delta V_T}{\partial f_2} \frac{f_1 - f_2}{\frac{\partial \Delta V_T}{\partial f_1} - \frac{\partial \Delta V_T}{\partial f_2}} .$$

This value, f_3, gives a partial derivative of ΔV_T with respect to f that is closer to zero than either $\partial \Delta V_T / \partial f_1$ or $\partial \Delta V_T / \partial f_2$. The nth calculation for f would be

$$f_n = f_{n-1} - \frac{\partial \Delta V_T}{\partial f_{n-1}} \frac{f_{n-2} - f_{n-1}}{\frac{\partial \Delta V_T}{\partial f_{n-2}} - \frac{\partial \Delta V_T}{\partial f_{n-1}}} .$$

Since $|f| \leq |\lambda|$ for optimality, if a predicted $|f|$ is greater than $|\lambda|$, $\bar{f}$ should be set to λ. If a second prediction for $|f|$ goes beyond $|\lambda|$, the two burn off-perilune deboost reduces to a one burn deboost for the assumed ρ_1. Thus when $\partial \Delta V_T / \partial f = 0$ exists, for any $\delta > 0$ a value of f can be obtained such that $|\partial \Delta V_T / \partial f| < \delta$; otherwise f is set equal to λ.

Carrying forward the assumption that ΔV_T is roughly parabolic with f (or ρ_1), insight into this two burn deboost can be gained from Figure 4. This figure indicates schematically the requirement for comparing $|f|$ with $|\lambda|$. When ΔV_T (f) is characteristic of Figure (4b), use of the predicted f will, in the case represented by (4b), result in instability in the iterative scheme.

We now proceed in a similar fashion to determine $\partial \Delta V_T / \partial \rho_1$. Using the value of f such that $|\partial \Delta V_T / \partial f| < \delta$, choose a second value for ρ_1 and then use the previously described method on ρ_1 and the partial of ΔV_T with respect to ρ_1 to find a value of ρ_1 such that $|\partial \Delta V_T / \partial \rho_1| < \delta$. Since $|\rho_1| \leq |i_2|$ for optimality, if a predicted value of $|\rho_1|$ exceeds $|i_2|$ set ρ_1 to i_2; and if a second $|\rho_1|$ exceeds $|i_2|$, the two burn off-perilune deboost reduces to a one burn deboost. For the value of ρ_1 such that $|\partial \Delta V_T/\partial \rho_1| < \delta$, the $|\partial \Delta V_T/\partial f|$ does not necessarily have to be less than δ.* Using this new value of ρ_1, f can be calculated by the iterative method such that $|\partial \Delta V_T/\partial f| < \delta$. By such successive calculations, a value can be obtained for ρ_1 and f such that $|\partial \Delta V_T/\partial f| < \delta$ and $|\partial \Delta V_T/\partial \rho_1| < \delta$.

For values of λ less than about 20°, suggested initial conditions for ρ_1 and f are i_2 and λ respectively, as this initialization isolates immediately the cases where the single burn is optimum. For larger values of λ, initial conditions of $\rho_1 = f = i_2 \left(1 - \frac{\lambda}{90}\right)$ have been observed to result in favorable convergence.

THE MAXIMUM LUNAR ACCESSIBILITY PROBLEM

When calculating the maximum lunar area accessible for a fixed ΔV_T**, i_2 assumes its maximum value. (This assertion is proven by inspection of Figure 1. For any value of λ, the spherical area between the α-plane and the β_2-plane is maximized for the maximum value of i_2. This spherical area, in turn, is just the lunar accessible area.) An equation analogous to Equation (1) may be formulated for the accessibility problem:

$$i_2 = F_2 \, (f, \rho_1, \lambda). \tag{24}$$

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*Recall that although ρ_1 and f are independent, ΔV_T is dependent upon both.

**The techniques for such calculations are documented in Reference 4. By this method, the ΔV allowed for getting into lunar parking orbit is assumed. The ΔV and propellant required to get out of the parking orbit and onto a transearth trajectory can then be calculated. By iterative techniques the parking orbit is adjusted such that the propellant requirements match the propellant availability.

Since ρ_1, f and λ are independent the necessary conditions that i_2 possess a maximum or minimum for a specified ΔV_T are then,

$$\frac{\partial F_2}{\partial \rho_1} = 0, \quad \frac{\partial F_2}{\partial f} = 0, \quad \frac{\partial F_2}{\partial \lambda} = 0. \qquad (25, 26, 27)$$

In this case it is convenient to assume a fixed λ and carry out a determination of the maximum i_2 with respect to ρ_1 and f. This assumption results in no loss of generality as will be discussed later. Equations (4) through (6) are applicable to this problem. A fourth equation can be obtained from the law of sines

$$\frac{\sin \rho_1}{\sin \psi} = \frac{\sin \rho_2}{\sin (\lambda - f)}, \text{ or}$$

$$\sin \rho_1 \sin (\lambda - f) = \sin \rho_2 \sin \psi. \qquad (28)$$

A fifth equation can be obtained from two equations from spherical trigonometry (Reference 5, page 189 and Reference 3, page 12).

$$\cos \rho_2 = \cos i_2 \cos \rho_1 + \sin i_2 \sin \rho_1 \cos (\lambda - f) \qquad (29)$$

and

$$-\cos (\lambda - f) \cos i_2 = \sin (\lambda - f) \cot \psi - \sin i_2 \cot \rho_1. \qquad (30)$$

Solving the first for $\sin i_2$, substituting this in the second equation, and multiplying both sides of the equation by $\sin \rho_1$ $\cos (\lambda - f)$ gives

$$- \cos^2 (\lambda - f) \cos i_2 \sin \rho_1 = \sin (\lambda - f) \cos (\lambda - f) \cot \psi \sin \rho_1$$

$$- \cos \rho_2 \cot \rho_1 + \cos i_2 \cos \rho_1 \cot \rho_1 . \quad (31)$$

The necessary sixth and seventh equations, satisfying conditions of Equations (25) and (26), may now be determined. Equations (32) through (35) develop the necessary equations for $\partial i_2/\partial \rho_1$ and $\partial i_2/\partial f$.

The partial derivative of i_2 with respect to ρ_1 is formed as follows. The partial derivative of Equation (28) with respect to ρ_1 is

$$\cos \rho_2 \sin \psi \frac{\partial \rho_2}{\partial \rho_1} + \sin \rho_2 \cos \psi \frac{\partial \psi}{\partial \rho_1} = \cos \rho_1 \sin (\lambda - f). \quad (32)$$

Take the partial of Equation (29) with respect to ρ_1 and solve for $\partial \rho_2/\partial \rho_1$.

$$\frac{\partial \rho_2}{\partial \rho_1} = \Bigg[\sin i_2 \cos \rho_1 \frac{\partial i_2}{\partial \rho_1} + \cos i_2 \sin \rho_1$$

$$- \cos i_2 \sin \rho_1 \cos (\lambda - f) \frac{\partial i_2}{\partial \rho_1}$$

$$- \sin i_2 \cos \rho_1 \cos (\lambda - f) \Bigg] / \sin \rho_2 .$$

The partial of Equation (30) with respect to ρ_1 solved for $\frac{\partial \psi}{\partial \rho_1}$ is

$$\frac{\partial \psi}{\partial \rho_1} = \left[- \cos i_2 \cot \rho_1 \frac{\partial i_2}{\partial \rho_1} + \sin i_2 \csc^2 \rho_1 - \cos (\lambda - f) \sin i_2 \frac{\partial i_2}{\partial \rho_1} \right] / (\sin (\lambda - f) \csc^2 \psi).$$

Substituting these last two equations back into Equation (32) gives

$$\frac{\cos \rho_2 \sin \psi}{\sin \rho_2} \left[\sin i_2 \cos \rho_1 \frac{\partial i_2}{\partial \rho_1} + \cos i_2 \sin \rho_1 - \cos i_2 \sin \rho_1 \cos (\lambda - f) \frac{\partial i_2}{\partial \rho_1} - \sin i_2 \cos \rho_1 \cos (\lambda - f) \right]$$

$$+ \frac{\sin \rho_2 \cos \psi}{\sin (\lambda - f) \csc^2 \psi} \left[- \cos i_2 \cot \rho_1 \frac{\partial i_2}{\partial \rho_1} + \sin i_2 \csc^2 \rho_1 - \cos (\lambda - f) \sin i_2 \frac{\partial i_2}{\partial \rho_1} \right] = \cos \rho_1 \sin (\lambda - f).$$

Solving this equation for $\partial i_2 / \partial \rho_1$,

$$\frac{\partial i_2}{\partial \rho_1} = \frac{G}{H} \qquad (33)$$

where

$$G = \cot \rho_2 \sin \psi \ (- \cos i_2 \sin \rho_1 + \sin i_2 \cos \rho_1 \cos (\lambda - f))$$

$$- \frac{\sin \rho_2 \cos \psi \sin^2 \psi \sin i_2}{\sin (\lambda - f) \sin^2 \rho_1} + \cos \rho_1 \sin (\lambda - f)$$

and

$$H = \cot \rho_2 \sin \psi \ (\sin i_2 \cos \rho_1 - \cos i_2 \sin \rho_1 \cos (\lambda - f))$$

$$- \frac{\sin \rho_2 \cos \psi \sin^2 \psi}{\sin (\lambda - f)} \ (\cos i_2 \cot \rho_1 + \cos (\lambda - f) \sin i_2),$$

gives an expression for the partial of i_2 with respect to ρ_1 and supplies the sixth equation.

The partial of i_2 with respect to f is formed in a very similar manner. The partial of Equation (28) with respect to f is

$$\cos \rho_2 \sin \psi \frac{\partial \rho_2}{\partial f} + \sin \rho_2 \cos \psi \frac{\partial \psi}{\partial f} = - \sin \rho_1 \cos (\lambda - f). \tag{34}$$

The partial derivative of Equation (29) with respect to f, solved for $\partial \rho_2 / \partial f$, and the partial of Equation (30) with respect to f, solved for $\partial \psi / \partial f$, are

$$\frac{\partial \rho_2}{\partial f} = \left[\sin i_2 \cos \rho_1 \frac{\partial i_2}{\partial f} - \cos i_2 \sin \rho_1 \cos (\lambda - f) \frac{\partial i_2}{\partial f} \right.$$

$$\left. - \sin i_2 \sin \rho_1 \sin (\lambda - f) \right] / \sin \rho_2,$$

and

$$\frac{\partial \psi}{\partial f} = \left[\sin(\lambda - f) \cos i_2 - \cos(\lambda - f) \sin i_2 \frac{\partial i_2}{\partial f} \right.$$

$$\left. - \cos(\lambda - f) \cot \psi - \cos i_2 \cot \rho_1 \frac{\partial i_2}{\partial f} \right] / (\sin(\lambda - f) \csc^2 \psi).$$

Substituting these equations into (34)

$$\frac{\cos \rho_2 \sin \psi}{\sin \rho_2} \left[\sin i_2 \cos \rho_1 \frac{\partial i_2}{\partial f} - \cos i_2 \sin \rho_1 \cos(\lambda - f) \frac{\partial i_2}{\partial f} \right.$$

$$\left. - \sin i_2 \sin \rho_1 \sin(\lambda - f) \right] + \frac{\sin \rho_2 \cos \psi}{\sin(\lambda - f) \csc^2 \psi}$$

$$\left[\sin(\lambda - f) \cos i_2 - \cos(\lambda - f) \sin i_2 \frac{\partial i_2}{\partial f} \right.$$

$$\left. - \cos(\lambda - f) \cot \psi - \cos i_2 \cot \rho_1 \frac{\partial i_2}{\partial f} \right] = - \sin \rho_1 \cos(\lambda - f)$$

and solving for $\frac{\partial i_2}{\partial f}$,

$$\frac{\partial i_2}{\partial f} = \frac{D}{E} \tag{35}$$

where

$$D = -\left[\cot \rho_2 \sin \psi \sin i_2 \sin \rho_1 \sin (\lambda - f) - \sin \rho_1 \cos (\lambda - f)\right.$$

$$\left. + \sin \rho_2 \cos \psi \sin^2 \psi \,(\cot (\lambda - f) \cot \psi - \cos i_2)\right]$$

and

$$E = \cot \rho_2 \sin \psi \;(\cos i_2 \sin \rho_1 \cos (\lambda - f) - \sin i_2 \cos \rho_1)$$

$$+ \frac{\sin \rho_2 \cos \psi \sin^2 \psi}{\sin (\lambda - f)} \;(\cos (\lambda - f) \sin i_2 + \cos i_2 \cot \rho_1),$$

we obtain the necessary seventh equation, an expression for the partial of i_2 with respect to f.

The variables are ΔV_1, ΔV_2, ρ_1, ρ_2, f, ϕ, and i_2. λ must be assumed for a solution but would normally be varied through 360° to generate a total lunar accessibility map. Then the known quantities are λ and ΔV_T. If ρ_1 and f are assumed ΔV_1, ΔV_2, and ρ_2 can be calculated from Equations (5), (4) and (6), respectively. From Equations (28) and (31), ψ and i_2 can be calculated as

$$\psi = \arcsin (\sin \rho_1 \sin (\lambda - f) / \sin \rho_2), \quad 0° \leq \psi \leq 90° \qquad (36)$$

$$i_2 = \arccos \left[\frac{\cos \rho_2 \cot \rho_1 - \sin (\lambda - f) \cos (\lambda - f) \cot \psi \sin \rho_1}{\cos \rho_1 \cot \rho_1 + \cos^2 (\lambda - f) \sin \rho_1} \right], \qquad (37)$$

$$0 \leq i_2 < 180°,$$

and ϕ can be calculated from Equation (22).

Substituting values for ΔV_1, ρ_1, ρ_2, ψ, f and i_2 back in Equations (33) and (35), values for $\partial i_2/\partial \rho_1$ and $\partial i_2/\partial f$ can be calculated and checks for optimality can be made. Depending on these values, new values for ρ_1 and f can be obtained by a recursive technique such that $\partial i_2/\partial \rho_1$ and $\partial i_2/\partial f$ are both sufficiently close to zero.

GENERALIZATIONS

It will now be shown that the equations which have been developed for the acute spherical triangle of Figure 1 are valid for all values of i_2, λ, and ϕ. General values of λ and ϕ are illustrated by Cases 2, 3, and 4 of Figures 5a and 5b. Figure 5a shows the two cases where $(\lambda-f)$ is positive, and Figure 5b shows the two cases for which $(\lambda-f)$ is negative. Case #1 and Case #3 place the spacecraft into the same lunar parking orbit, the only difference being that the second burn occurs half an orbit later in Case #3; inspection indicates that the optimum values of ρ_1 and $|\rho_2|$ are the same for both cases. This same relationship holds between Case #2 and Case #4. Hence, the equations which were developed for Case #1 [$(\lambda-f) \geq 0$ and $\rho_2 \geq 0$] need only be validated for Case #2 [$(\lambda-f) < 0$ and $\rho_2 < 0$] to be completely general.

For the minimum ΔV_T problem, ψ is not involved in the solution and ϕ is determinable and unambiguous over the range $0° \leq \phi \leq 180°$ from the arctan function of Equation (22). Notice that negative values of $(\lambda-f)$ result in a sign change in ρ_2 (Equation (23)). Thus for the minimum ΔV_T problem, Case #2 is distinguishable by the sign of $(\lambda-f)$.

For the maximum accessibility problem, ρ_2 must take its sign from $(\lambda-f)$, ψ must be placed in the second quadrant (Equation (36)) if $(\lambda-f) < 0$, and the quadrant of i_2 is determinable and unambiguous from the arccos function (Equation (37)).

It may aid the reader to note that the second plane change is always made in a manner which increases the inclination of the orbital plane. Any other maneuver would be non-optimum. Thus, the direction of the second plane change is uniquely determined by the sign of $(\lambda-f)$.

Figures 5a and 5b show only cases where ρ_1 and i_2 are positive (measured counter-clockwise). The cases where ρ_1

and i_2 are negative are mirror images about the α-plane in Figures 5a and 5b. The equations which have been developed remain unchanged when i_2 and ρ_1 are negative and are so treated in the trignometric formulae. It should be noted that negative values of ρ_1 are uniquely associated with negative i_2.

QUANTITATIVE RESULTS FROM THE MINIMUM ΔV_T PROBLEM

Figure 6 illustrates the potential ΔV savings inherent in the two burn deboost; it presents ΔV_T as a function of λ and i_2 for the one burn deboost; the optimum two burn deboost; and, for comparison, the two burn deboost for which the first burn is constrained to occur at perilune. For all calculations, V_H was assumed as 8300 fps and V_C as 5300 fps. An initial inspection of the figure indicates that constraining the first burn to occur at perilune generally incurs significant costs for values of i_2 greater than about 4° and that the additional complexity involved in optimizing on f is worthwhile.

The major point to be drawn from Figure 6 is the comparison between the one burn and two burn deboost. Relatively, the larger savings are realized from cases of small i_2 and $\lambda > 10°$. Of course, the two burn technique results in significant economies for large values of i_2 but here the relative efficiencies begin at about $\lambda = 25°$. As the figure shows, the savings are quite sensitive to λ; that is, the two burn deboost is particularly appropriate where λ exceeds 15°. Because perilune on free return trajectories occurs near 180° selenographic longitude, large values (greater than 15°) of λ will be most prevalent for lunar landing sites situated near 0° longitude with latitudes on the order of 10°. (CSM orbital plane change requirements constrain the lunar parking orbit to azimuths near 270° at the lunar landing site.) Hence, this analysis gives further verification that the greatest propellant savings from this two burn deboost will occur for landing sites near 0° longitude.

SUMMARY

Analytic solutions to the optimum "two burn" deboost into lunar parking orbit have been developed. For this particular two burn technique, the first burn transfers (with a plane change) from the lunar approach hyperbola to an intermediate circular lunar parking orbit and the second burn transfers to the final required lunar parking orbit by means

of the appropriate plane change. The equations required for optimization are developed for two general cases: the minimum total ΔV needed to reach a specified lunar parking orbit and the maximum lunar accessibility available from a fixed ΔV. The optimization is two dimensional in that both the optimum point of transfer from the approach hyperbola and the optimum distribution of plane change between the two burns are determined. By necessity, one additional iteration loop is required as compared with the single burn technique; however, this added iteration is in the nature of an "inner loop" and additional computation time is commensurate. For the minimum total ΔV problem an iterative technique is described which is based on the orbital geometry and equations involved.

Finally, quantitative results are presented which reaffirm that the greatest efficiencies from this two burn technique occur for lunar landing sites in the central region (near 0° longitude).

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2013-SFC-srb

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Attachments:
References
Table I
Figures 1-6

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TABLE I

Glossary of Symbols

a	semi-major axis of lunar approach hyperbola
e	eccentricity of lunar approach hyperbola
f	true anomaly of first burn
i_2	inclination of the final orbit plane to the hyperbolic approach plane
V_C	velocity on the circular lunar parking orbit
V_H	velocity on lunar approach hyperbola at first burn (circular parking orbit altitude)
ΔV_1	ΔV required for first burn
ΔV_2	ΔV required for second burn
ΔV_T	total ΔV required for two burn deboost
α-plane	plane of lunar approach hyperbola
β_1-plane	lunar orbit plane after first burn
β_2-plane	lunar parking orbit plane after second burn - desired final orbit plane
γ	flight path angle at first burn
ρ_1	plane change during first burn
ρ_2	plane change during second burn
ϕ	great circle angle from first burn to second burn measured in the β_1-plane
λ	true anomaly of node between the α-plane and the β_2-plane
ψ	great circle angle from the α-plane/β_2-plane node to the β_1-plane/β_2-plane node

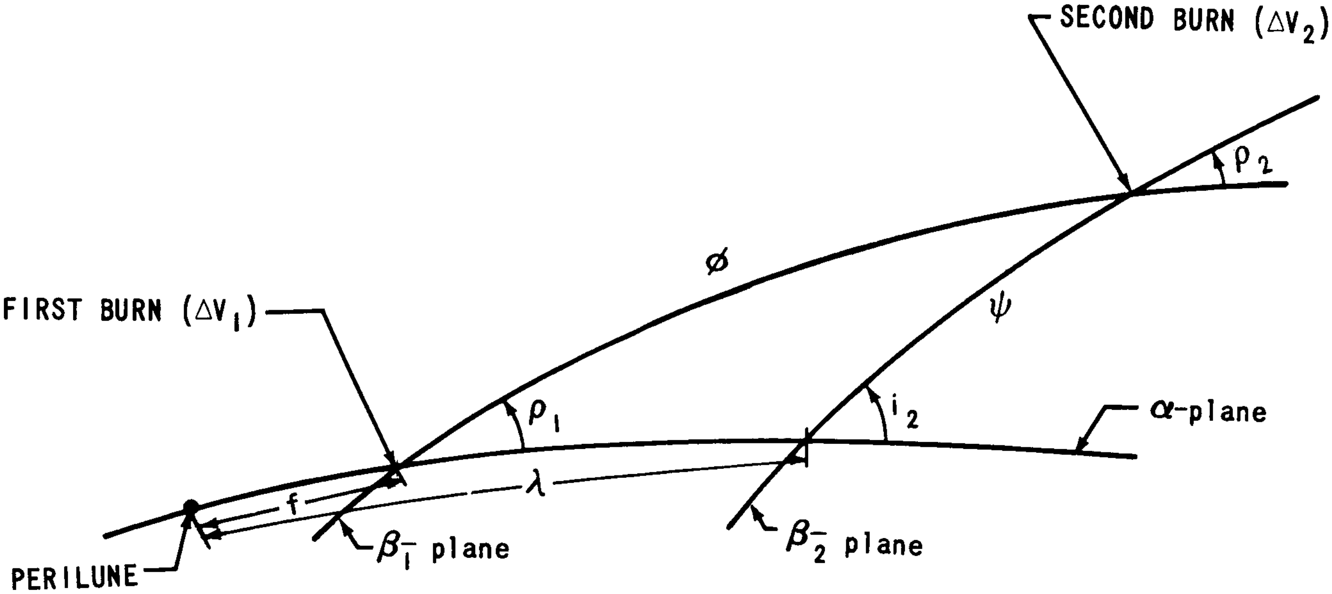

FIGURE 1 - SPHERICAL GEOMETRY OF THE TWO BURN DEBOOST

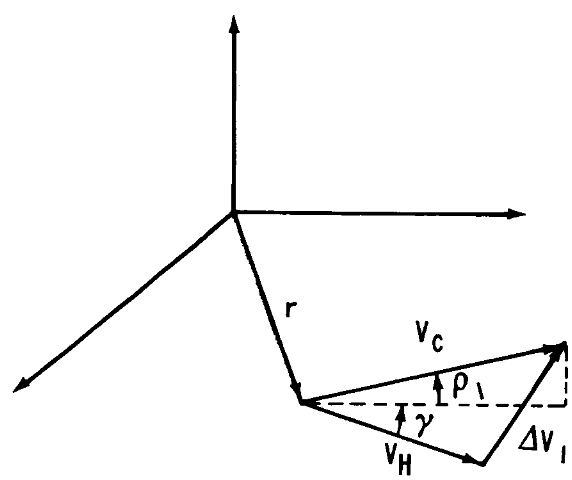

FIGURE 2 - GEOMETRY OF THE FIRST BURN

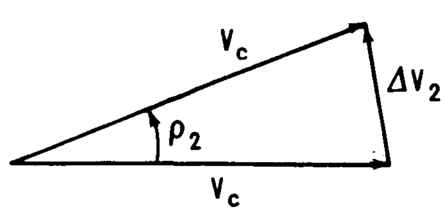

FIGURE 3 - GEOMETRY OF THE SECOND BURN

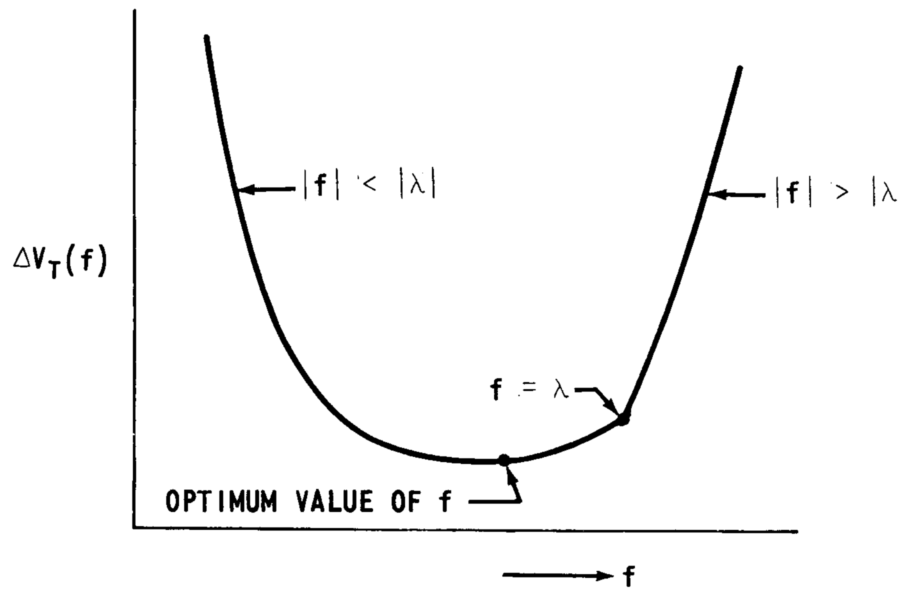

FIGURE 4a - TWO BURN DEBOOST MORE EFFICIENT THAN ONE BURN DEBOOST

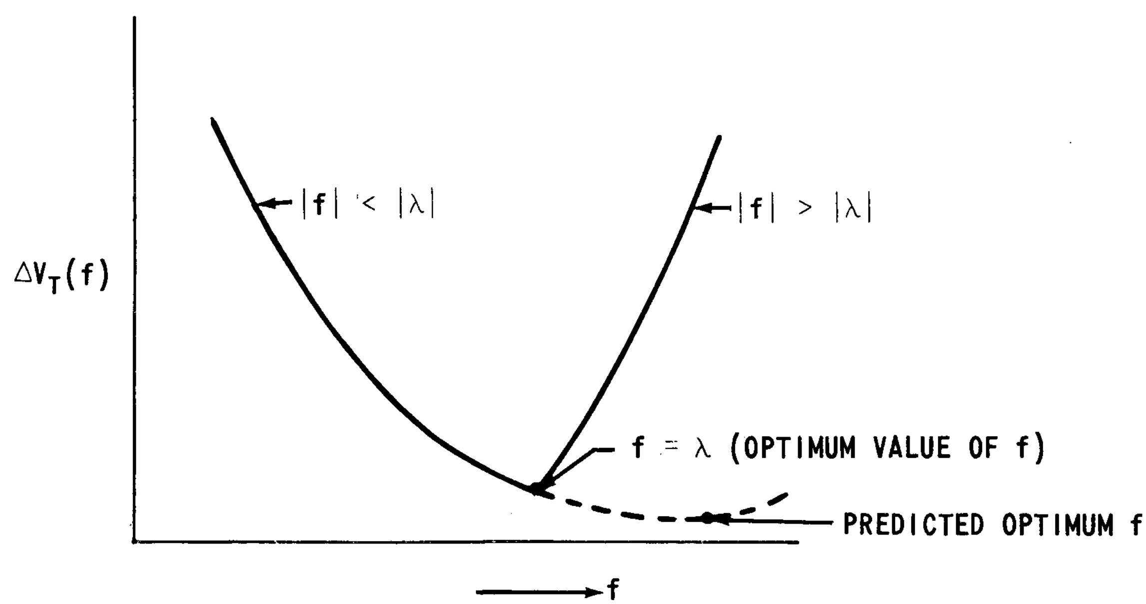

FIGURE 4b - OPTIMUM TWO BURN DEBOOST DEGENERATES TO SINGLE BURN

FIGURE 4 - SCHEMATIC REPRESENTATION OF THE FUNCTIONAL NATURE OF ΔV_T, f, AND λ

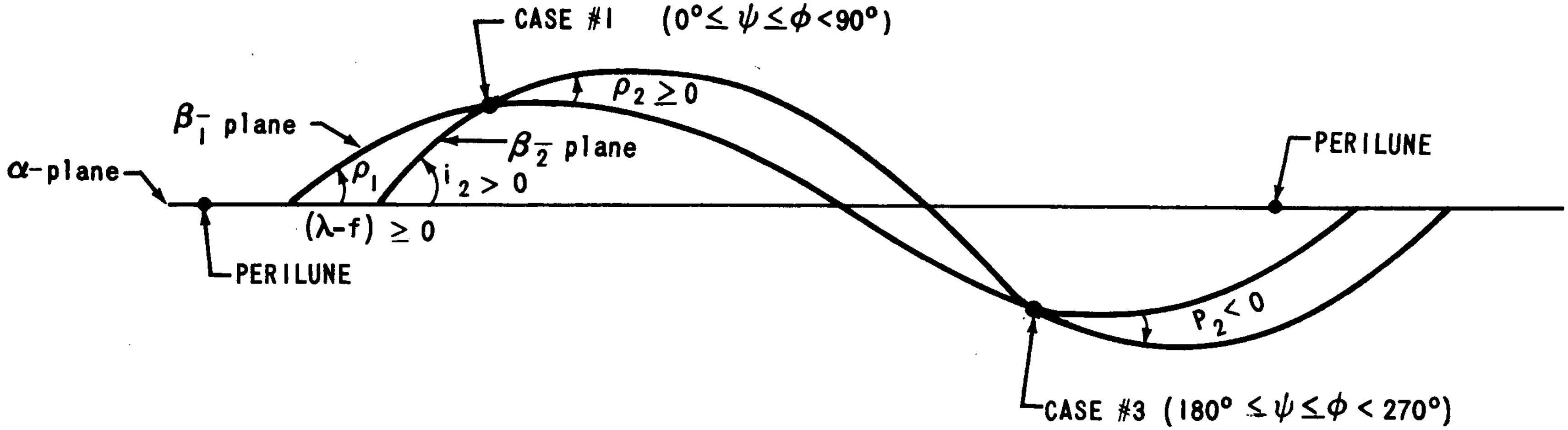

FIGURE 5a: $(\lambda - f) \geq 0$

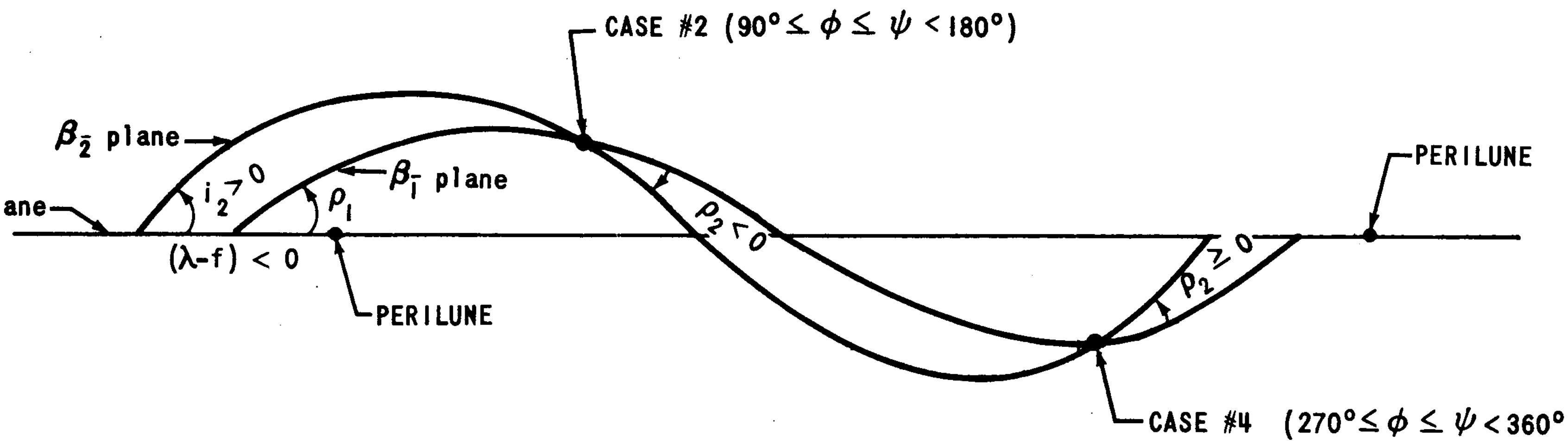

FIGURE 5b: $(\lambda - f) < 0$

FIGURE 5 - GENERALIZED GEOMETRY FOR THE TWO BURN DEBOOST

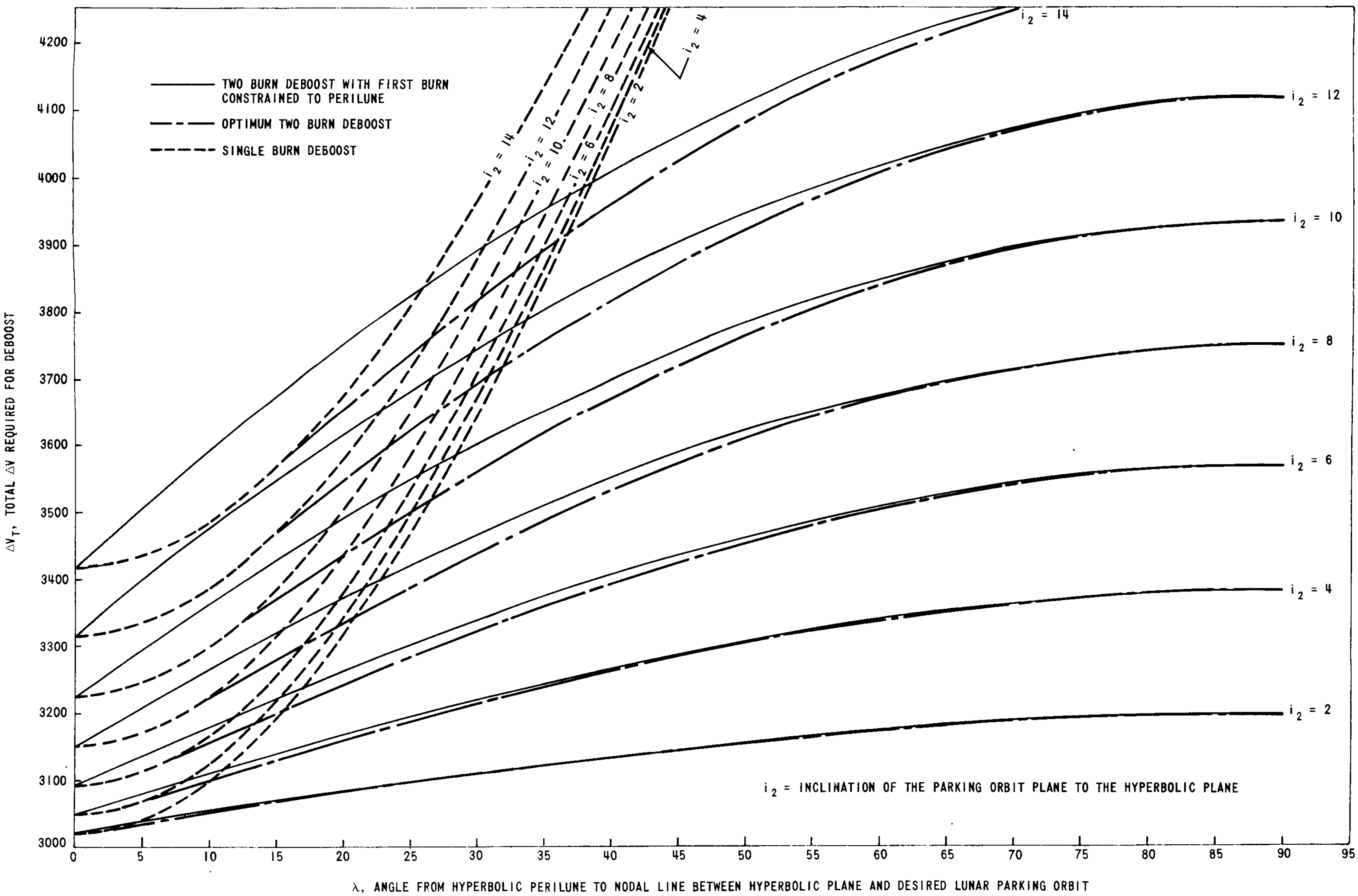

FIGURE 6 - TOTAL DEBOOST REQUIREMENTS AS A FUNCTION OF HYPERBOLIC PLANE AND LUNAR ORBIT GEOMETRY